Mathematics 207 Real Analysis

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The purpose of real analysis is to provide a rigorous foundation for the techniques of calculus, which are based on the notion of *limit*. The exercises assume familiarity with the basic ideas of *convergence of a sequence* of real numbers and the definition of *continuity of a function* in terms of the standard symbols $\varepsilon > 0$ and $\delta > 0$ along with the definition of *derivative*. We also assume the *Fundamental theorem of Calculus* and take for granted the *integrability of any continuous function*. The known nature of the *real numbers* is assumed, including the existence of the *greatest lower bound* of a set bounded below and similarly the *least upper bound* of a set bounded above. Set 1 establishes the elementary properties of convergent sequences of real numbers. Set 2 is concerned with certain limits that are especially important, particularly those involving the number e. Set 3 introduces results and examples on continuity of a function. Throughout we will work mainly with one variable mappings although we occasionally expand to matters of several variables. Sets 4 and 5 concern series. We introduce and work with the standard tests for convergence and examples include the *binomial series* for non-integral powers. We draw on all this knowledge in the second part of the module.

In Set 6 we study continuous functions on closed intervals (the prototype of so-called *compact sets*, which we shall meet in Level 3 modules in a more general setting). We prove the *Intermediate value* and *Extreme value* theorems for continuous functions on a closed interval and illustrate the ideas involved with relevant examples. Set 7 introduces the concept of *uniform continuity* for individual and for sequences of functions. This condition is key in justifying many of the techniques of calculus that involve the interchange of limiting operations, such as *term-by-term differentiation and integration* of series. In Set 8 we study *power series* where the uniform convergence of the series within its *radius of convergence* is a crucial property in calculations involving power series representation of functions of interest. In particular the *Weierstrass M-test* is a tool we first meet here. Set 9 introduces and proves another fundamental result of calculus, that being the *Mean value theorem* in various forms and we use the MVT to prove theorems often used in calculus including *Equality of mixed partial derivates*. Set 9 and all of Set 10 are about *Taylor series* and we introduce a study of the *Remainder term* both in the *Lagrange form*, based on the Mean value theorem, and the *Integral form*. We close with some practical calculations including a brief visit into the realm of Taylor series of several variables.

Solutions and Comments for the Problems

Problem Set 1

1. Suppose to the contrary that $M < A$. Put $\varepsilon = A - M > 0$. Since $a_n \to A$ there exists N such that for all $n \geq N$,

$$|A - a_n| < \varepsilon$$

$$\Rightarrow A - a_n \leq |A - a_n| < A - M$$

$$\Rightarrow a_n > M,$$

a contradiction, and so $\lim_{n\to\infty} a_n \leq M$, the given upper bound of the sequence.

2. Given $\varepsilon > 0$ taken N_1, N_2 such that $|a_n - A| < \varepsilon$ for all $n \geq N_1$ and $|b_n - B| < \varepsilon$ for all $n \geq N_2$. Put $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ we have by the triangle inequality:

$$|\lambda a_n + \mu b_n - (\lambda A + \mu B)| = |\lambda(a_n - A) + \mu(b_n - B)|$$

$$\leq |\lambda||a_n - A| + |\mu||b_n - B| \leq |\lambda|\varepsilon + |\mu|\varepsilon = \varepsilon(|\lambda| + |\mu|)$$

and since $|\lambda| + |\mu|$ is a fixed constant, it follows that $(\lambda a_n + \mu b_n) \to \lambda A + \mu B$.

Comment We can end the argument with ε rather than a multiple of ε if we wish by taking $|a_n - A| < \frac{\varepsilon}{|\lambda|+|\mu|}$ etc. (while also dealing with the trivial case where $\lambda = \mu = 0$). It is a matter of taste whether or not to introduce such a contrivance in order to satisfy the formal definition of convergence.

3. (a) Let A be the limit of the sequence $(a_n)_{n\geq 1}$ and put $\varepsilon = 1$. Then there exists N such that for all $n \geq N$ we have $|a_n - A| < 1$. Then for any $n \geq N$ we have

$$|a_n| = |a_n - A + A| \leq |a_n - A| + |A| \leq |A| + 1 \tag{1}$$

Next let $B = \max\{|a_n| : n \leq N - 1\}$. Then for all $n \geq 1$ we have

$$|a_n| \leq M =: \max\{B, 1 + |A|\},$$

and so $(a_n)_{n\geq 1}$ is bounded.

Comment The conclusion may be written as $-M \leq a_n \leq M$ so that the sequence itself has both a lower and an upper bound.

(b) Any convergent sequence is bounded above and below by 3(a). Conversely, suppose that $(a_n)_{n\geq 1}$ is a monotonic increasing sequence that is bounded above. (The argument in the decreasing case is the same except for the direction of the inequalities involved.) Since $(a_n)_{n\geq 1}$ is bounded above, the sequence has a ***least upper bound*** (also known as the ***supremum***) A and we claim that

$a_n \to A$. Too see this, let $\varepsilon > 0$ be given. Then there exists N such that $A - \varepsilon < a_N \leq A$ for if there were no such N, then $A - \varepsilon$ would be an upper bound of the sequence that was less than the least upper bound, which is a contradiction. Then, since $(a_n)_{n\geq 1}$ is increasing in n, it follows that for any $n \geq N$ we have $A - \varepsilon \leq a_N \leq a_n \leq A$ and in particular $|A - a_n| < \varepsilon$ for all $n \geq N$. Therefore it follows that $a_n \to A$, as required.

(c) Let $\varepsilon > 0$ be given and take N such that for all $n \geq N$ we have $|a_n - A| < \varepsilon$. Then by the Triangle inequality we have

$$||a_n| - |A|| \leq |a_n - A| \leq \varepsilon,$$

whence it follows that $|a_n| \to |A|$.

(d) The converse is false: for example let $a_n = (-1)^n$. Then $|a_n| = 1$ so that $(|a_n|)_{n\geq 1} \to 1$ but the sequence $(a_n)_{n\geq 1}$ has no limit at all.

4. By Question 3 there exists a common positive upper bound M for the convergent sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$. Similarly, for any given $\varepsilon > 0$, there is a common index N such that for all $n \geq N$ we have $|a_n - A| < \varepsilon$ and $|b_n - B| < \varepsilon$. Then

$$|a_n b_n - AB| = |a_n b_n - Ab_n + Ab_n - AB| \leq |(a_n - A)b_n + A(b_n - B)|$$

$$\leq |a_n - A||b_n| + |A||b_n - B| \leq \varepsilon M + |A|\varepsilon = \varepsilon(M + |A|),$$

which is a constant multiple of ε and so we conclude that $a_n b_n \to AB$.

5. It is enough to prove this in the case where a_n is the constant sequence 1, for given this and Question 3 we have

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \to A \cdot \frac{1}{B} = \frac{A}{B}.$$

Now

$$|\frac{1}{b_n} - \frac{1}{B}| = |\frac{B - b_n}{Bb_n}| \tag{2}$$

Choose N such that for all $n \geq N$, $|b_n - B| < |B|\varepsilon$ and $|b_n| \geq \frac{1}{2}|B|$ (so that $\frac{1}{|b_n|} \leq \frac{2}{|B|}$). To prove that the latter is possible first we note by Question 3(c) that $|b_n| \to |B|$. Take $\varepsilon = \frac{|B|}{2} > 0$. Then we may take N such that for all $n \geq N$, $||b_n| - |B|| < \varepsilon$ so that

$$-\varepsilon < |b_n| - |B| < \varepsilon$$

$$\Rightarrow |b_n| > |B| - \frac{|B|}{2} = \frac{|B|}{2}.$$

Then for all $n \geq N$ we have by (4) that

$$|\frac{1}{b_n} - \frac{1}{B}| \leq \frac{|B|\varepsilon}{|B||b_n|} = \frac{\varepsilon}{|b_n|} \leq \frac{2\varepsilon}{|B|},$$

from which follows that $\frac{1}{b_n} \to \frac{1}{B}$, as required.

Comment Even without the condition that $b_n \neq 0$ for all $n \geq 1$ we have that the convergence of the tail of the sequence $\frac{a_n}{b_n}$ to the limit $\frac{A}{B}$ still holds as $b_n \to B \neq 0$ implies that only finitely many of the b_n can equal 0, and we may simply consider the behaviour of the sequence after the point where there are no further zero values in the b_n .

6. Let $\varepsilon > 0$ and take N such that for all $n \geq N$, $|a_n - A| < \varepsilon$. For the sequence $(a_{n_i})_{i\geq 1}$ take j such that $n_j \geq N$. Then for any $i \geq j$ we have $n_i \geq n_j \geq N$ so that $|a_{n_i} - A| < \varepsilon$ and so it follows that $a_{n_i} \to A$.

7. Take N such that for all $n \geq N$, $|a_n - l| \leq \frac{\varepsilon}{2}$. Then for any $m, n \geq N$ we obtain the required inequality as follows:

$$|a_m - a_n| = |a_m - l - (a_n - l)| \leq |a_m - l| + |a_n - l| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

8. We have that some interval $I_0 = [-M, M]$ contains all members a_n of our sequence. It follows that at least one of the intervals $[-M, 0]$ and $[0, M]$ contains infinitely many members of the sequence. Choose such an interval I_1 and repeat the argument, splitting I_1 into two closed intervals of equal length with common endpoint. In this way we define a nested sequence of intervals

$$I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

with $|I_n| = \frac{M}{2^{n-1}}$. We then form a subsequence $(a_{n_i})_{i\geq 0}$ by choosing $a_{n_i} \in I_i$.

Now let $\varepsilon > 0$ and take $i \in \mathbb{Z}^+$ such that $\frac{M}{2^{i-1}} < \varepsilon$. Take any $j, k \geq i$. Then since $a_{n_j}, a_{n_k} \in I_i$ we have

$$|a_{n_j} - a_{n_k}| \leq \frac{M}{2^{i-1}} < \varepsilon,$$

which shows that the subsequence $(a_{n_i})_{i\geq 0}$ of $(a_n)_{n\geq 0}$ is Cauchy convergent. Hence, by the completeness of $\mathbb{R}$, it follows that $(a_{n_i})_{i\geq 0}$ converges, as required to complete the proof.

Comment We shall take the results of the previous questions, and simple consequences thereof, for granted in future proofs without explicit reference. Another point to note is that the convergence or otherwise of a sequence is unaltered if we adjoin or omit a finite number of terms.

9. Let $\varepsilon > 0$. Take N such that for all $n \geq N$, $|a_n - A| < \frac{\varepsilon}{2}$ and let M be an upper bound for $(|a_n|)_{n\geq 1}$. Then for any $n \geq N$ such that $\frac{MN}{n} < \frac{\varepsilon}{2}$ we have

$$|\frac{1}{n}\sum_{k=1}^{n} a_k - A| = |\frac{1}{n}\sum_{k=1}^{N} a_k + \frac{1}{n}\sum_{k=N+1}^{n} a_k - A|$$

$$\leq \frac{1}{n}|\sum_{k=1}^{N} a_k| + \frac{1}{n}|(\sum_{k=N+1}^{n} a_k - A)|$$

$$\leq \frac{MN}{n} + \frac{1}{n}|\sum_{k=N+1}^{n} (a_k - A)|$$

$$\leq \frac{MN}{n} + \frac{1}{n}\sum_{k=N+1}^{n} |a_k - A|$$

$$\leq \frac{\varepsilon}{2} + \frac{n(\varepsilon/2)}{n} = \varepsilon.$$

10. We have

$$S_n = \frac{1}{\lfloor \frac{n}{2} \rfloor}\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} k(n-k)$$

for $n = 2m$ we have, using standard formulas for the sums of powers, that

$$\Rightarrow \frac{S_n}{n^2} = \frac{1}{4m^3}(\frac{2m^2(m+1)}{2} - \frac{m(m+1)(2m+1)}{6})$$

$$= \frac{1}{24m^2}(6m(m+1) - (2m+1)(m+1)) = \frac{1}{24m^2}(4m-1)(m+1)$$

$$= \frac{1}{24}(4 - \frac{1}{m})(1 + \frac{1}{m}) \rightarrow \frac{4}{24} = \frac{1}{6}.$$

For $n = 2m + 1$ we have

$$\frac{S_n}{n^2} = \frac{1}{m(2m+1)^2}(\frac{(2m+1)m(m+1)}{2} - \frac{m(m+1)(2m+1)}{6})$$

$$= \frac{1}{6(2m+1)^2}(2(2m+1)(m+1)) = \frac{1}{3}(\frac{m+1}{2m+1}) = \frac{1}{3}(\frac{1+\frac{1}{m}}{2+\frac{1}{m}})$$

$$\rightarrow \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}.$$

Problem Set 2

1.

$$e(n) = (\frac{1}{n} + 1)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!n^k}$$

and the term indexed by k is given by

$$t(k) = \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n}). \tag{3}$$

We observe that $t_k > 0$ and is increasing in n as this is true of each of the factors. Also $e(n+1)$ has one more term than does $e(n)$, whence it follows that $2 \leq e(n) < e(n+1)$.

2(a) Replacing each bracketed term of $t(k)$ by 1 we see that

$$e(n) < \sum_{k=0}^{n} \frac{1}{k!} \tag{4}$$

(b) Then observe that $2^{k-1} < k!$ for all $k \geq 1$ so that $\frac{1}{k!} < \frac{1}{2^{k-1}}$; we obtain:

$$e(n) < 1 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1-(1/2)^n}{1-(1/2)} = 1 + \frac{2^n - 1}{2^{n-1}}$$

$$= 1 + 2 - \frac{1}{2^{n-1}} < 3.$$

3. The McLaurin series for e^x is given by

$$e^x = \sum_{k=0}^{\infty} \frac{(e^x)^{(n)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and putting $x = 1$ then gives:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \tag{5}$$

4. If $m > n$ we have

$$e(m) > 1+1+\frac{1}{2!}(1-\frac{1}{m})+\frac{1}{3!}(1-\frac{1}{m})(1-\frac{2}{m})+\cdots+\frac{1}{n!}(1-\frac{1}{m})\cdots(1-\frac{n-1}{m}) \tag{6}$$

as $e(m)$ is comprised of the sum on the right hand side of (8) together with more positive terms (see (5) above). Letting $m \to \infty$ then gives that for all $n \geq 0$

$$e = \lim_{m\to\infty} e(m) \geq s(n) =: \sum_{k=0}^{n} \frac{1}{k!} \tag{7}$$

On the other hand by (6) we have $e(n) \leq s_n$. Hence we have $e(n) \leq s_n \leq e$; letting $n \to \infty$ we then have $e = \lim_{n\to\infty} s_n$. Therefore we have equality throughout and arrive at two equivalent definitions for the number e:

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{k=0}^{\infty}\frac{1}{k!}.$$

Comment The expression $e(n)$ was first introduced in *Bernoulli's compound interest problem*, which asks for the limiting amount of interest gained when interest accrues continuously. Looked at this way, it is clear that $(1+\frac{1}{n})^n$ is increasing in n as this expression represents the interest accruing when interest is paid at n equally spaced intervals per annum (and interest rate is 100%) and interest on interest will accrue earlier if interest is paid more often.

5(a) Let $l(x) = \log_b(x)$ $(x > 0)$. Then for any $a > 0$ we have $l(\frac{x}{a}) = l(x) - l(a)$. Differentiating both sides by gives:

$$\frac{l'(\frac{x}{a})}{a} = l'(x)$$

putting $x = a$ then gives

$$l'(a) = \frac{l'(1)}{a};$$

or using the symbol x instead of a:

$$(\log_b(x))' = \frac{\lambda}{x},\ \lambda = (\log_b(x))'|_{x=1}.$$

(b) Hence we have

$$\lambda = \lim_{h\to 0} \frac{\log_b(1+h) - \log_b(1)}{h} = \lim_{h\to 0} \log_b(1+h)^{1/h} = \log_b(\lim_{h\to 0}(1+h)^{\frac{1}{h}}),$$

where we have assumed that the limit and the taking of log may be interchanged (which is valid because of the continuity of the log function). Putting $n = h^{-1}$ we get

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = \lim_{n\to\infty}(1+\frac{1}{n})^n = e.$$

Therefore $\lambda = \log_b e = 1$ if and only if $b = e$. This shows in particular that $(\ln x)' = x^{-1}$.

6. For $n = 2$ we have $(1+h)^2 = 1 + 2h + h^2 > 1 + 2h$ as $h \neq 0$. Suppose the claim holds for some $n \geq 2$ and consider

$$(1+h)^{n+1} = (1+h)^n(1+h) > (1+nh)(1+h) = 1+(n+1)h+h^2 > 1+(n+1)h,$$

and so the induction continues, thus completing the proof.

7. Since $p > 1$, we have $p^{1/n} > 1$ so that $a_n = 1 + h_n$ for some $h_n > 0$. Hence by Question 6,

$$p = (1+h_n)^n > 1 + nh_n$$

$$\Rightarrow 0 < h_n < \frac{p-1}{n} \to 0$$

$$\Rightarrow a_n \to 1 + 0 = 1.$$

Otherwise for $0 < p < 1$ we have that $0 < p^{\frac{1}{n}} < 1$ and so $p^{\frac{1}{n}} = 1 - r_n$ for some $0 < r_n < 1$. We seek to write this as $p^{\frac{1}{n}} = \frac{1}{1+h_n}$ so we solve

$$\frac{1}{1+h_n} = 1 - r_n$$

$$\Leftrightarrow h_n = \frac{1}{1-r_n} - 1 = \frac{r_n}{1-r_n}$$

and since $0 < r_n < 1$ it follows that $0 < h_n$, as we require. Since $(1+h_n)^n > 1 + nh_n$ we get that $(1+h_n)^{-n} < (1+nh_n)^{-1}$ and so

$$p = \frac{1}{(1+h_n)^n} < \frac{1}{1+nh_n}$$

$$\Rightarrow 1 + nh_n < \frac{1}{p}$$

$$\Rightarrow 0 < h_n < \frac{\frac{1}{p} - 1}{n} \to 0.$$

Therefore in the case where $0 < p < 1$ it also follows that $a_n = \frac{1}{1+h_n} \to 1$.

8. Note that $b_n^n = (n^{\frac{1}{2n}})^n = \sqrt{n}$ so that

$$\sqrt{n} = (1+h_n)^n > 1 + nh_n$$

$$\Rightarrow h_n < \frac{\sqrt{n} - 1}{n} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

9. We now have

$$1 \leq a_n = b_n^2 = 1 + 2h_n + h_n^2 \leq 1 + \frac{2}{\sqrt{n}} + \frac{1}{n} \to 1$$

and so

$$a_n = \sqrt[n]{n} \to 1, \text{ as } n \to \infty.$$

10. Put $a_n = \frac{n}{\alpha^n}$ so that $\sqrt{a_n} = \frac{\sqrt{n}}{(\sqrt{\alpha})^n}$. Since $\alpha > 1$, so is $\sqrt{\alpha}$ and so we may write $\sqrt{\alpha} = 1 + h$ for some $h > 0$. Then

$$\sqrt{\alpha^n} = (1+h)^n > 1 + nh, \text{ so that}$$

$$\sqrt{a_n} = \frac{\sqrt{n}}{(1+h)^n} \leq \frac{\sqrt{n}}{1+nh} \leq \frac{\sqrt{n}}{nh} = \frac{1}{h\sqrt{n}}$$

$$\therefore 0 < a_n =: \frac{n}{\alpha^n} \leq \frac{1}{nh^2} \to 0.$$

Problem Set 3

1. Let $\varepsilon > 0$ and take $\delta > 0$ such that $|x-l| < \delta$ implies that $|f(x)-f(l)| < \varepsilon$, which is possible as $f(x)$ is continuous at $x = l$. Take N such that for all $n \geq N$, $|a_n - l| < \delta$. Then $|f(a_n) - f(l)| < \varepsilon$, and therefore $(f(a_n))_{n\geq 1} \to f(l)$.

Comment Note this is saying that $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$, or in words, the actions of taking the limit and acting a continuous function on a convergent sequence may be interchanged.

2. For any $a \in \mathbb{R}$ including $a = 0$ we may also put $\delta = \varepsilon > 0$. For $a = 0$ if $|x - a| = |x| \leq \varepsilon$ then $||x| - |0|| = |x| \leq \varepsilon$ and so $|x|$ is continuous at $x = 0$. For $a \neq 0$ assume without loss that ε is chosen sufficiently small so that $|x - a| < \varepsilon$ implies that x and a have the same sign. Then for $a > 0$ we have $||x|-|a|| = |x-a| < \varepsilon$ while for $a < 0$ we have $||x|-|a|| = |-x+a| = |x-a| < \varepsilon$. In either case, this serves to show that $|x|$ is continuous for all $a \in \mathbb{R}$.

3. Since $g(x)$ is continuous at $x = f(a)$ it follows that for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|f(x) - f(a)| < \delta_1$ implies that $|g(f(x) - gf(a)| < \varepsilon$. Since $f(x)$ is continuous at $x = a$ if follows that there exists $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \delta_1$. Therefore for any x such that $|x - a| \leq \delta$ we obtain

$$|f(x) - f(a)| < \delta_1 \Rightarrow |g(f(x)) - g(f(a))| < \varepsilon,$$

thus proving that $g(f(x))$ is continuous at $x = a$.

By Question 2 we know that $|x|$ defines a continuous function so that by what we have just proved (putting $g(x) = |x|$) we have that if $f(x)$ is continuous then so is $|f(x)|$.

To see that the converse is false as we may take $f(x)$ to be the function that takes the value 1 if $x \in \mathbb{Q}$ and -1 if $x \notin \mathbb{Q}$. Then $|f(x)| \equiv 1$, the constant function 1, which is clearly continuous, yet $f(x)$ is not a continuous function. Indeed $f(x)$ is discontinuous at every point as each point has arbitrarily small neighbourhoods where the function values of points within the neighbourhood differ by 2.

4. Given $\varepsilon > 0$, let $\delta_1, \delta_2 > 0$ be such that $|x - a| < \delta_1$ implies that $|f(x) - f(a)| < \varepsilon$ and $|x - a| < \delta_1$ implies that $|g(x) - g(a)| < \varepsilon$. Put $\delta = \min\{\delta_1, \delta_2\}$. Then $|x - a| < \delta$ implies that

$$|h(x) - h(a)| = |\lambda(f(x) - f(a)) + \mu(g(x) - g(a)|$$

$$\leq |\lambda||f(x) - f(a)| + |\mu||g(x) - g(a)| \leq (|\lambda| + |\mu|)\varepsilon$$

and since λ and μ do not depend on ε, it follows that $h(x) = \lambda f(x) + \mu g(x)$ is continuous at $x = a$.

5. Let $a \in \mathbb{R}$. Then

$$\sin(a+h) - \sin a = \sin a \cos h - \cos a \sin h - \sin a = \sin a(\cos h - 1) - \cos a \sin h.$$

Given that $\lim_{h\to 0} \cos h = 1$ and $\lim_{h\to 0} \sin x = 0$ we obtain:

$$\lim_{h\to 0}(\sin(a+h)-\sin a) = \sin a \lim_{h\to 0}(\cos h-1)-\cos a(\lim_{h\to 0} \sin h) = \sin a(1-1)-\cos a(0) = 0.$$

6. It suffices to prove the case where $p(n) = n^k$ for some $k \geq 1$, with the $k = 1$ case being dealt with in Question 10 Set 2. We proceed by induction on k. Let $k \geq 2$ and let $b = \sqrt{a} > 1$. Then by induction and the $k = 1$ case we obtain:

$$\lim_{n\to\infty} \frac{n^k}{a^n} = \lim_{n\to\infty} \frac{n^{k-1}}{b^n} \cdot \frac{n}{b^n} = \lim_{n\to\infty} \frac{n^{k-1}}{b^n} \cdot \lim_{n\to\infty} \frac{n}{b^n} = 0 \cdot 0 = 0.$$

7(a) We are given that

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

Define the function $\varepsilon(h)$ by the equation

$$\varepsilon(h) = \frac{f(a+h) - f(a)}{h} - f'(a)$$

$$\Rightarrow \lim_{h\to 0} \varepsilon(h) = f'(a) - f'(a) = 0;$$

$$\therefore f(a+h) - f(a) = hf'(a) + h\varepsilon(h)$$

$$\Rightarrow \lim_{h\to 0}(f(a+h) - f(a)) = \lim_{h\to 0} hf'(a) + \lim_{h\to 0} h\varepsilon(h) = 0 + 0 = 0.$$

Therefore $f(a+h) \to f(a)$ as $h \to 0$, which is to say that $f(x)$ is continuous at $x = a$.

(b)

$$f'(x) = \lim_{h\downarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\uparrow 0} \frac{f(x-h) - f(x)}{-h} = -\lim_{h\uparrow 0} \frac{f(x-h) - f(x)}{h} \text{ and}$$

$$f'(x) = \lim_{h\uparrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\downarrow 0} \frac{f(x-h) - f(x)}{-h} = -\lim_{h\uparrow 0} \frac{f(x-h) - f(x)}{h}.$$

Since a limit exists if and only if each of the corresponding two-sided limits exist and agree, it follows that we may also define

$$f'(x) = -\lim_{h\to 0} \frac{f(x-h) - f(x)}{h}$$

and vice-versa.

(c) Suppose that $f(x)$ is even (and differentiable). Then, making use of (b), we have

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} = -f'(x)$$

so that $f'(x)$ is odd. On the other hand if $f(x)$ were odd then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h} = -\lim_{h\to 0} \frac{f(x-h) - f(x)}{h} = f'(x),$$

thus showing that $f'(x)$ is even.

8. Using polar coordinates we have $x^2 + y^2 = r^2$ and $x^3 = r^3 \cos^3\theta$. The required limit then takes the form:

$$\lim_{r\to 0} \frac{r^3 \cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0$$

as $|\cos^3\theta| \leq 1$ independently of the value of θ. Hence if we define $f(0,0) = 0$ the function $f(x, y)$ is continuous throughout all of the domain $\mathbb{R}^2$.

9. Putting $y = mx$ the limit takes the form:

$$\lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{x^2(1-m^2)}{x^2(1+m^2)} = \frac{1-m^2}{1+m^2};$$

since the limit is not constant but rather its value depends on the gradient of the line of approach to the origin, it follows that no single limiting value may be assigned to $f(0,0)$ that makes the function continuous at the origin.

10(a) Again putting $y = mx$ gives the limit:

$$\lim_{x\to 0} \frac{mx^3}{x^4 + m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2 + m^2};$$

if $m \neq 0$, this limit is $\frac{0}{m^2} = 0$. If $m = 0$ (i.e. we approach along the line $y = 0$) we also get $\lim_{x\to 0} \frac{0}{x^4} = 0$.

(b) However if we approach the origin along the curve $y = x^2$, the limit exists but takes on a different value:

$$\lim_{x\to 0} \frac{x^4}{x^4 + x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Problem Set 4

1. Let $s_n = \sum_{k=1}^n a_k$. We have that $s_n \to S$, where S is the sum of the series. The sequence $(s_n)_{n\geq 1}$ is Cauchy convergent and in particular for any $\varepsilon > 0$ there exists N such that for all $n \geq N$, $|s_{n+1} - s_n| < \varepsilon$, which is to say $|a_{n+1}| < \varepsilon$ for all $n \geq N$. Since ε was arbitrary it follows that $a_n \to 0$.

2. Suppose that Σ converges so that $s_n \to S$ say. Let us write $t_{n,m}$ for $\sum_{k=n+1}^{k=n+m} a_n = s_{n+m} - s_n$. Then for any $\varepsilon > 0$ there exists N such that for all $n \geq N$, $|s_n - s| < \frac{\varepsilon}{2}$. Hence for all $m \geq 0$ we have

$$|s_{n+m} - s| = |t_{n,m} - s + s_n)| < \frac{\varepsilon}{2}.$$

$$\Rightarrow -\frac{\varepsilon}{2} < t_{n-m} + (s_n - s) < \frac{\varepsilon}{2}$$

$$\Rightarrow -\varepsilon < t_{n,m} < \varepsilon \Leftrightarrow |t_{n,m}| < \varepsilon \, \forall m \geq 0$$

$$\Rightarrow \lim_{m\to\infty} |t_{n,m}| =: t_n = |\sum_{k=n+1}^{\infty} a_n| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary it follows that $|t_n| \to 0$ as $n \to \infty$, so the same is true of t_n, i.e.

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_n = 0. \tag{8}$$

Conversely suppose that (8) is true. Let $\varepsilon > 0$ and take N such that for all $n \geq N$, $|t_n| \leq \frac{\varepsilon}{2}$. Then for any $m \geq 0$ we have

$$-\frac{\varepsilon}{2} < \sum_{k=n+1}^{\infty} a_n = s_{n+m} - s_n + \sum_{k=n+m+1}^{\infty} a_n < \frac{\varepsilon}{2}$$

$$-\varepsilon < s_{n+m} - s_n < \varepsilon,$$

and so for all $n \geq N$ and $m \geq 0$, $|s_{n+m} - s_n| < \varepsilon$, and since ε was arbitrary it follows that $\sum_{k=0}^{\infty} a_n$ converges.

3. We have that $\sum_{n=1}^{\infty} |a_n|$ converges so given any $\varepsilon > 0$ there exists N such that for all $n \geq N$, and $m \geq 1$, $\sum_{k=n+1}^{n+m} |a_k| \leq \varepsilon$. Hence, by the Triangle inequality we obtain:

$$|\sum_{k=n+1}^{n+m} a_k| \leq \sum_{k=n+1}^{n+m} |a_k| \leq \varepsilon$$

$$\Leftrightarrow |s_{n+m} - s_n| \leq \varepsilon$$

and so $\sum_{k=1}^{\infty} a_n$ converges.

4. We show inductively that the series $(s_{2n})_{n\geq 0}$ and $(s_{2n+1})_{n\geq 0}$ are respectively monotonically decreasing and monotonically increasing. Suppose that we have $s_0 \geq s_2 \geq \cdots \geq s_{2n}$ for some $n \geq 0$ (the $n = 0$ case being vacuously true). Then

$$s_{2(n+1)} = s_{2n+2} = s_{2n} + (a_{2n+2} - a_{2n+1})$$

and by hypothesis the bracketed term is non-positive (as $a_{2n+2} \leq a_{2n+1}$) and so $s_{2n} \geq s_{2(n+1)}$ and the induction continues. Similarly suppose we have $s_1 \leq s_3 \leq \cdots \leq s_{2n+1}$ for some $n \geq 0$, the base case again being clear. Then

$$s_{2(n+1)+1} = s_{2n+3} = s_{2n+1} + (a_{2n+2} - a_{2n+1}) \geq s_{2n+1},$$

and the induction continues; therefore $(a_{2n+1})_{n\geq 0}$ is decreasing. The claim is thus established.

Next we observe that $s_{2n} \geq s_{2m+1}$ for any $n, m \geq 0$. To see this, suppose to the contrary that for some n, m we have $s_{2m+1} > s_{2n}$. Take $k > m, n$. It follows from the claim that

$$s_{2k+1} \geq s_{2m+1} > s_{2n} \geq s_{2k}.$$

This gives $s_{2k+1} = s_{2k} - a_{2k+1} > s_{2k}$, a contradiction. Therefore we conclude that $s_{2n} \geq s_{2m+1}$ for all $n, m \geq 0$.

The sequence $(s_{2n})_{n\geq 0}$ is monotonic decreasing and is bounded below by all the s_{2n+1} so converges to a limt A, while similarly $(s_{2n+1})_{n\geq 0}$ is a monotonic increasing sequence bounded above by all the s_{2n} (and so by their limit A) and so converges to a limit B; it follows that $B \leq A$. We complete the proof by showing that $A = B$.

Suppose to the contrary that $B < A$ so we may write $A = B + \varepsilon$ for some $\varepsilon > 0$. Since $a_n \to 0$ it follows that the same is true of both of the subsequences $(a_{2n})_{n\geq 0}$ and $(a_{2n+1})_{n\geq 0}$. Take N such that for any $n \geq N$, $a_n < \frac{\varepsilon}{2}$. Then

$$(s_{2n} \geq A) \Rightarrow (s_{2n+1} = s_{2n} - a_{2n+2} \geq A - \frac{\varepsilon}{2} = B + \frac{\varepsilon}{2});$$

however B is the least upper bound of the sequence $(s_{2n+1})_{n\geq 0}$, and in particular B is an upper bound, and that is contradicted by $s_{2n+1} > B$. Therefore $A = B$ is the limit of the sequence $(s_n)_{n\geq 0}$.

5. For $p > 0$, the function $f(x) = x^{-p}$ is monotonically decreasing for $x \geq 1$, and so we have

$$I = \int_1^\infty \frac{dx}{x^p} > \sum_{n=2}^{\infty} \frac{1}{n^p}.$$

For $p > 1$ we have

$$I = \frac{x^{1-p}}{1-p}\Big|_1^\infty = 0 - \frac{1}{1-p} = \frac{1}{p-1}$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < 1 + \frac{1}{p-1} = \frac{p}{p-1}.$$

Since the sequence of partial sums of this series is monotonic increasing and bounded above, the series converges.

On the other hand if $p < 1$ we may observe that:

$$\sum_{n=1}^{N} \frac{1}{n^p} > \int_1^N \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}|_1^N = \frac{N^{1-p}-1}{1-p}$$

and since the latter expression approaches infinity as $N \to \infty$, it follows that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p < 1$. If $p = 1$ we obtain

$$\sum_{n=1}^{N} \frac{1}{n} > \int_1^N \frac{dx}{x} = \ln x|_1^N = \ln N \to \infty.$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

6. Suppose that $r < 1$. Let $2\varepsilon = 1 - r > 0$. Note that $s = r + \varepsilon = 1 - \varepsilon < 1$. There exists N such that

$$-\varepsilon < |\frac{a_{n+1}}{a_n}| - r < \varepsilon \ \forall n \geq N$$

$$\Rightarrow 0 < |a_{n+1}| < s|a_n|$$

$$\Rightarrow |a_{n+1}| < s^{n-N}|a_N| \ \forall n \geq N.$$

Let s_N denote $\sum_{n=0}^{N} |a_n|$. Then

$$\sum_{n=0}^{\infty} |a_n| = s_N + \sum_{n=N+1}^{\infty} |a_n| \leq s_N + |a_N| \sum_{n=N+1}^{\infty} s^{n-N}$$

$$= s_N + |a_N| \sum_{n=1}^{\infty} s^n = s_N + |a_N| \frac{s}{1-s}.$$

Therefore since the partial sums s_n of the series $(|a_n|)_{n \geq 1}$ are monotonically increasing and bounded above, it follows that the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, and so convergent.

Next suppose that $r > 1$. Take $\varepsilon > 0$ such that $s = r - \varepsilon > 1$. Then take N such that for all $n \geq N$, $|\frac{a_{n+1}}{a_n}| > s$. Hence we have $|a_{n+m}| > s^m |a_n|$. In particular $\lim_{n\to\infty} a_n \neq 0$, whence it follows by Question 1 that the series $\sum_{n=0}^{\infty} a_n$ is divergent.

7. $e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}$. In this case

$$|\frac{a_{n+1}}{a_n}| = |\frac{n! x^{n+1}}{(n+1)! x^n}| = \frac{|x|}{n+1} \to 0 \ \forall x \in \mathbb{R}$$

and so by the Ratio test, the series converges for all x.

$\sin x \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\cos x \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. The test ratios are respectively:

$$|\frac{(-1)^{n+1}(2n+1)!x^{2(n+1)+1}}{(-1)^n(2(n+1)+1)!x^{2n+1}}| = |\frac{x^2}{(2n+2)(2n+3)|}| \to 0 \ \forall x \in \mathbb{R}$$

$$|\frac{(-1)^{n+1}(2n)!x^{2(n+1)}}{(-1)^n(2(n+1))!x^{2n}}| = |\frac{x^2}{(2n+1)(2n+2)|}| \to 0 \ \forall x \in \mathbb{R}$$

and so, by the Ratio test, both these series also coverge for all real x.

8. Note that $s_N = \sum_{n=1}^{N} a_n$ is a strictly monotonic increasing sequence in N and that $s_N \leq \sum_{n=1}^{N} b_n \leq B = \sum_{n=1}^{\infty} b_n$. Therefore $S = \sum_{n=1}^{\infty} a_n$ converges (to a limit no more than B). On the other hand if the series S is divergent then so is $\sum_{n=1}^{\infty} b_n$, for if this series were convergent then so would S be convergent (by the previous argument).

9.

$$\text{(i) } |\frac{a_{n+1}}{a_n}| = |\frac{2^{3(n+1)}}{(n+1)!} \cdot \frac{n!}{2^{3n}}| = |\frac{2^3}{n+1}| \to 0 \text{ so series converges;}$$

$$\text{(ii) } |\frac{a_{n+1}}{a_n}| = |\frac{2^{(n+1)^2}}{(2(n+1))!} \cdot \frac{(2n)!}{2^{n^2}}| = |\frac{2^{2n+1}}{(2n+2)(2n+1)}| \to \infty \text{ so series diverges;}$$

(iii) The Ratio test limit here is 1 so that test is inconclusive. However, for $n \geq 3$, $\frac{\ln n}{n} > \frac{1}{n}$ and since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges then so does $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

10.

$$\text{(i) } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{(1+n^2)^2}{1-2n^2} = \infty, \text{ so series diverges;}$$

$$\text{(ii) } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n}{5^{3+\frac{2}{n}}} = \infty, \text{ so series diverges;}$$

$$\text{(iii) } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{1+n}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{1+n}\right)^n = e^{-1} < 1, \text{ so series converges.}$$

Problem Set 5

1. From Question 3 of Set 1, we prove this by showing that one of these series is bounded above if and only if the other is as well. Let s_n and t_n denote the respective sequences of partial sums of the two series:

$$s_n = a_1 + a_2 + \cdots + a_n$$

$$t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}.$$

Since the a_n are decreasing, it follows that for $n \leq 2^k$,

$$s_n \leq a_1 + (a_2 + a_3) + \cdots + (a_{2^{k-1}} + a_{2^{k-1}+1} + \cdots + a_{2^k - 1}) + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})$$

$$\leq a_1 + 2a_2 + \cdots + 2^{k-1} a_{2^{k-1}} + 2^k a_{2^k} = t_k,$$

so that $s_n \leq t_k$ for $n \leq 2^k$. On the other hand, if $n \geq 2^k$,

$$s_n \geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}} + a_{2^{k-1}+1} + \cdots + a_{2^k})$$

$$\geq \frac{a_1}{2} + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} = \frac{t_k}{2}$$

so that $2s_n \geq t_k$. It follows that the sequences $(s_n)_{n \geq 1}$ and $(t_n)_{n \geq 1}$ are both bounded above, or both not bounded above, and therefore the corresponding series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2^n a_{2^n}$ both converge or both diverge.

2. Let $a_n = n^{-p}$ $(p \neq 1)$. Then the sequence consists of positive monotonically decreasing terms and so we may apply Question 1. Applying the ratio test to $(2^n a_{2^n})_{n \geq 1}$ gives in this case

$$\frac{2^{n+1}(2^{n+1})^{-p}}{2^n(2^n)^{-p}} = \frac{2 \cdot 2^{np}}{2^{(n+1)p}} = \frac{2}{2^p} = \frac{1}{2^{p-1}};$$

now if $p > 1$ then $p - 1 > 0$ and the ratio is less than 1, telling us that the series in question both converge. On the other hand if $p < 1$ then $p - 1 < 0$ and the ratio exceeds 1, indicative of divergent series.

3. Here we have $a_n = \frac{1}{n(\log n)^p}$, which is a monotonic decreasing sequence of positive terms and so we may apply Cauchy condensation and instead look at the sum of the condensed series:

$$\sum_{n=2}^{\infty} \frac{2^n}{2^n(\log(2^n))^p} = \sum_{n=2}^{\infty} \frac{1}{n^p(\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{n=2}^{\infty} \frac{1}{n^p};$$

and by Question 2, we know this series converges if and only if $p > 1$.

4. Since the terms of the series are positive and monotonically decreasing we may apply the integral test to $\sum_{n=2}^{\infty} \frac{1}{n \log n(\log(\log n))}$ and so consider the corresponding integral:

$$I = \int_2^{\infty} \frac{dx}{x \log x(\log(\log x))}.$$

Put $u = \log(\log x)$. Then $du = \frac{dx}{x \log x}$ so we get:

$$I = \int_{\log(\log 2)}^{\infty} \frac{du}{u} = [\log u]_{\log(\log 2)}^{\infty},$$

which is infinite, and so the series in question also diverges.

5(a) Suppose to the contrary that f were not continuous at u, whence there exists some $\varepsilon > 0$ such that for any $\delta > 0$ there exists $x \in S$ such that $|x-u| < \delta$ but $|f(x)-f(u)| > \varepsilon$. In particular we may choose $u_n \in S$ such that $|x_n-u| < \frac{1}{n}$ but $|f(u_n) - f(u)| > \varepsilon$. But then $(u_n)_{n\geq 1}$ is a sequence in S converging to u but for all n we have $|f(u_n) - f(u)| > \varepsilon$. Hence if f is not continuous at u there exists a sequence in S that converges to u but the sequence of images, $f(u_n)$, does not converge to $f(u)$. By the contrapositive, we conclude that if every sequence in S that converges to u has its image sequence converging to $f(u)$, then f is continuous at u.

(b) The converse is also true for suppose that f is continuous at u and let $(u_n)_{n\geq 1}$ be a sequence in S that converges to u. Let $\varepsilon > 0$. Since f is continuous at u, there exists $\delta > 0$ such that if $x \in S$ with $|x-u| < \delta$ then $|f(x)-f(u)| < \varepsilon$. Then there exists N such that for all $n \geq N$ we have $|u_n - u| < \delta$, whence $|f(u_n) - f(u)| < \varepsilon$, thereby showing that $(f(u_n))_{n\geq 1} \to f(u)$, as required.

6(a) We note that

$$\frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}, \text{ hence}$$

$$\sum_{n=1}^{N} \frac{n}{(n+1)!} = \sum_{n=1}^{N} \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = \frac{1}{1!} - \frac{1}{(N+1)!} = 1 - \frac{1}{(N+1)!}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \lim_{N\to\infty} \left(1 - \frac{1}{(N+1)!}\right) = 1.$$

(b)

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \Rightarrow \frac{e^x-1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

$$\Rightarrow \frac{xe^x - e^x + 1}{x^2} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!}$$

$$\Rightarrow \frac{1 - e^x + xe^x}{x} = \sum_{n=1}^{\infty} \frac{nx^n}{(n+1)!}.$$

We now put $x = 1$ and so obtain the same result:

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \frac{1-e+e}{1} = 1.$$

7(a) Differentiating $f(x) = (1+x)^\alpha$ gives $f'(x) = \alpha(1+x)^{\alpha-1}$, whence

$$(1+x)f'(x) = \alpha(1+x)^\alpha = \alpha f(x).$$

(b) Write $f(x) = \sum_{n=0}^{\infty} a_n x^n$ so that our equation takes on the form:

$$(1+x)f'(x) = (1+x)\sum_{n=1}^{\infty} n a_n x^{n-1} = (1+x)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \alpha \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow (n+1)a_{n+1} + n a_n = \alpha a_n \ \forall n \geq 0$$

$$\Rightarrow a_{n+1} = \frac{\alpha - n}{n+1} a_n = \frac{(\alpha-n)(\alpha-n+1)}{(n+1)n} a_{n-1} = \frac{(\alpha-n)(\alpha-n+1)(\alpha-n+2)}{(n+1)n(n-1)} a_{n-2} =$$

$$\cdots = \frac{(\alpha-n)(\alpha-n+1)\cdots(\alpha-1)\alpha}{(n+1)n(n-1)\cdots 2\cdot 1} a_0$$

and since $a_0 = f(0) = 1$ we conclude, upon replacing $n+1$ by n in the preceding calculation, that

$$a_n = \frac{(\alpha-n+1)(\alpha-n+2)\cdots\alpha}{n!} \ \forall n \geq 1.$$

Note that $a_0 = 1$, $a_1 = \alpha$, $a_2 = \frac{(\alpha-1)\alpha}{2}, \cdots$.

8. We apply the ratio test:

$$|\frac{a_{n+1}x^{n+1}}{a_n x^n}| = |\frac{(\alpha-n)(\alpha-n+1)\cdots\alpha}{(n+1)!} \cdot \frac{n! x}{(\alpha-n+1)(\alpha-n+2)\cdots\alpha}| = \frac{|\alpha-n||x|}{n+1} \to |x|;$$

hence the series converges if $|x| > 1$ and diverges if $|x| < 1$.

9. Since

$$\phi(x) = \frac{f(x)}{(1+x)^{\alpha}} \Rightarrow \phi'(x) = \frac{f'(x)(1+x)^{\alpha} - \alpha(1+x)^{\alpha-1}f(x)}{(1+x)^{2\alpha}}.$$

However this numerator can be worked using the equation $(1+x)f'(x) - \alpha f(x) = 0$ as follows:

$$= (1+x)f'(x)(1+x)^{\alpha-1} - \alpha f(x)(1+x)^{\alpha-1} = ((1+x)f'(x) - \alpha f(x))(1+x)^{\alpha-1} = 0.$$

Hence $\phi(x)$ is constant, and the value of that constant is

$$\phi(1) = \frac{a_0}{(1+0)^{\alpha}} = \frac{1}{1} = 1;$$

$$\therefore (1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \ \forall -1 < x < 1.$$

10. We have $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$ so that the general coefficient in the expansion takes on the form:

$$\binom{\frac{1}{2}}{n} = \frac{(\frac{1}{2}-n+1)(\frac{1}{2}-n+2)\cdots\frac{1}{2}}{n!}$$

$$\therefore \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots$$

Putting $x = 1$ gives:

$$\sqrt{2} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} = 1\frac{64-16+8-5}{128} = 1\frac{51}{128} = 1\cdot 40\,(2\,\text{d.p}).$$

Problem Set 6

1. Put $2\varepsilon = f(a) > 0$. Then since $\lim_{x\to a^+} f(x) = f(a)$, there exists $\delta > 0$ such that if $0 < x - a < \delta$ then $|f(x) - f(a)| < \varepsilon$, so that

$$-\varepsilon < f(x) - f(a) < \varepsilon$$

$$\Rightarrow \varepsilon = f(a) - \varepsilon < f(x) < f(a) + \varepsilon;$$

in particular $f(x) > 0$ for all x such that $0 \le x - a < \delta$.

Comment Similarly if $f(a) < 0$ we can find $\delta > 0$ such that $f(x) < 0$ for all $0 \le x - a < \delta$. Moreover, it is clear that the same holds in each case for a suitably chosen closed interval $[0, \delta]$.

2. Let $A = \{x : a \le x \le b,\ f(y) < 0\,\forall\, a \le y \le x\}$. Since $f(a) < 0$ we have that $A \ne \emptyset$. Since $f(b) > 0$ and $f(x)$ is continuous, there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in [b-\delta, b]$. Hence there exists a least upper bound α to A and $a \le \alpha < b$. We show that $f(\alpha) = 0$.

Suppose to the contrary that $f(\alpha) < 0$. There there exists $\delta > 0$ such that for all $x \in (\alpha - \delta, \alpha + \delta)$, $f(x) < 0$. Now there is some $x_0 \in A$ that satisfies $\alpha - \delta < x_0 < \alpha$ because otherwise α would not be the *least* upper bound of A. This means that f is negative on $[a, x_0]$. But then for any $x_1 \in [\alpha, \alpha + \delta)$ then f is negative on $[x_0, x_1]$. Then f is negative on $[a, x_1]$. This gives $x_1 \in A$ and $\alpha < x_1$, condtradicting that α is an upper bound of A. Hence the assumption that $f(\alpha) < 0$ must be false.

On the other hand, suppose that $f(\alpha) > 0$. Then, again by continuity, there exists $\delta > 0$ such that for all $x \in [\alpha - \delta, \alpha]$ we have $f(x) > 0$. But then $\alpha - \delta$ is a smaller upper bound for A than α, again a contradiction. Therefore $f(\alpha) = 0$. Since $f(a), f(b)$ are both non-zero we conclude that $a < \alpha < b$.

Comment By applying this argument to $-f$, it follows that the conclusion of the IVF also holds if $f(a) > 0$ and $f(b) < 0$.

3(i) Let $f(x) = x - \cos x$. Then $f(0) = 0 - 1 = -1 < 0$; $f(\frac{\pi}{2}) = \frac{\pi}{2} - 0 = \frac{\pi}{2} > 0$ and since f is continuous, by the Intermediate value theorem, there exists $x \in (0, \frac{\pi}{2})$ such that $f(x) = x - \cos x = 0$, which is to say that $x = \cos x$.

(ii) Let $f(x) = x - 1 - \sin x$. Then $f(0) = 0 - 1 - 0 = -1 < 0$ while $f(2) = 2 - 1 - \sin 2 = 1 - \sin 2 > 0$. Again by the IVT it follows that there exists $x \in (0, 2)$ such that $f(x) = x - 1 - \sin x = 0$, so that $\sin x = x - 1$.

(iii) Without loss we may take the leading coefficient of $p(x)$ to be 1, so that $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$, with n odd. Then for $x \neq 0$ we may write:

$$p(x) = x^n(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x}) + x^n.$$

By the IVF it is enough to show that $p(x)$ takes on values of both signs. Let A be the maximum of the numbers $|a_0|, |a_1|, \cdots, |a_{n-1}|, 1$. Then for any x such that $|x| \geq 2(n-1)A$ we have by the Triangle inequality that

$$|\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x}| \leq \frac{A}{2(n-1)A} + \frac{A}{2(n-1)A} + \cdots + \frac{A}{2(n-1)A} = \frac{1}{2}.$$

It follows that for any x such that $x \geq 2(n-1)A$ we have $p(x) \geq x^n - \frac{1}{2}x^n = \frac{1}{2}x^n$ and if $x < 2(n-1)A$ then $p(x) \leq x^n + \frac{1}{2}x^n$; in particular, $p(x) > 0$ if $x > 2(n-1)A$ and, since n is odd, $p(x) < 0$ if $x < -2(n-1)A$. It now follows the the IVT that $p(x)$ has a real root.

4(a) Let $a \in f^{-1}(U)$ so that $f(a) = u \in U$. Since U is open there exists $\varepsilon > 0$ such that if $|y - u| < \varepsilon$ then $y \in U$. Now since $f(x)$ is continuous there exists $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| = |f(x) - u| < \varepsilon$ so that $f(x) \in U$ and $x \in f^{-1}(U)$. This shows that the sphere of radius $\delta > 0$ centred at x lies in $f^{-1}(U)$ and since x was an arbitrary member of $f^{-1}(U)$ it follows that $f^{-1}(U)$ is open.

Conversely, suppose that for every open set $U \subseteq \mathbb{R}^m$,$f^{-1}(U)$ is open. Let $\varepsilon > 0$, let $a \in \mathbb{R}^n$, and consider the open sphere U of radius ε centred at $f(a)$. By hypothesis, $f^{-1}(U)$ is an open set, which contains a. Let $\delta > 0$ be such that for the sphere V of radius δ centred at a we have $V \subseteq f^{-1}(U)$. Then $f(V) \subseteq U$ so that if $b \in \mathbb{R}^n$ is such that $|b - a| < \delta$ then $|f(b) - f(a)| < \varepsilon$, thereby showing that $f(x)$ is continuous at the arbitrary point a.

(b) Yes, for it is equivalent to the result of part (a). Let $U \subseteq \mathbb{R}^m$ and let $U' = \mathbb{R}^m \setminus U$. Then $\mathbb{R}^n$ is a disjoint union of $f^{-1}(U)$ and $f^{-1}(U')$. Now by part (a) f is continuous if and only if $f^{-1}(U)$ is open for all open sets $U \subseteq \mathbb{R}^m$, which is equivalent to $f^{-1}(U')$ is closed for every closed set $U' \subseteq \mathbb{R}^m$.

5.

$$\lim_{h\to 0}((x+h)^2 - x^2) = \lim_{h\to 0} 2hx = 0;$$

and so $f(x) = x^2$ is continuous for all $x \in \mathbb{R}$.

However, take the open interval $I = (-1, 1)$. Then $f(I) = [0, 1)$, which is not open (as it contains the boundary point 0). Therefore a continuous map does not necessarily map open sets to open sets.

Comment An even simpler example of this kind is a constant mapping, which maps every set, open or otherwise, to the a one-point closed set. Similarly the sine function maps any subset of the real line that contains an interval of length 2π onto the closed interval $[-1, 1]$. It is possible to construct some (rather strange) mappings on the real line that do map open sets to open sets yet are not themselves continuous. A continuous mapping that does map open sets to open sets is called an *open mapping*.

6. Setting $\varepsilon = 1$ we may take $\delta > 0$ such that for all $x \in (a - \delta, a + \delta)$ $|f(x) - f(a)| < 1$. Therefore for all x in this interval we have

$$-1 < f(x) - f(a) < 1$$

$$\Rightarrow -1 + f(a) < f(x) < f(a) + 1$$

which gives lower and upper bounds for $f(x)$ on $(a - \delta, a + a\delta)$.

7. Let $A = \{x : a \leq x \leq b \text{ and } f \text{ is bounded on } [a, x]\}$. Then $a \in A$ and A is bounded above by b. Let α be the least upper bound of A. Suppose that $\alpha < b$. Then by Question 6, we have that f is bounded on some interval $(a - \delta, a + \delta)$ for some $\delta > 0$ (where, without loss, we may take δ sufficiently small so that $a + \delta \leq b$) and f is bounded on $[a, \alpha - \frac{\delta}{2}]$ (for otherwise $\alpha - \frac{\delta}{2}$ would be a smaller upper bound for A), whence it follows that f is bounded on the union of these two intervals, which is $[a, \alpha + \delta)$. However this contradicts that α is an upper bound for A. Therefore $\alpha = b$. Take $\delta < \frac{b-a}{2}$. It now follows that f is bounded on $[a, b - \delta]$.

By the same argument but with the interval $[a, x)$ replaced by $(x, b]$ in A, we conclude that f is bounded on $[a+\delta, b]$ and so f is bounded on $[a+\delta, b] \cup [a, b-\delta] = [a, b]$.

8. By Question 7 we have that $f(x)$ is bounded on $[a, b]$. Let M be the least upper bound of $f([a, b])$. For any $n \in \mathbb{Z}^+$ there exists $x_n \in [a, b]$ such that $f(x_n) > M - \frac{1}{n}$. Consider the sequence $(x_n)_{n \geq 1}$. Since $[a, b]$ is bounded, this sequence has a convergent subsequence $(x_{n_i})_{i \geq 1}$ with limit x say. Since $[a, b]$ is closed, this limit x is a member of $[a, b]$. We claim that $f(x) = M$. To see this we note that $\frac{1}{n_i} \leq \frac{1}{i} \to 0$ as $i \to \infty$.

Suppose that, contrary to our claim, that $f(x) = M - \varepsilon$ for some $\varepsilon > 0$. Choose i such that $\frac{1}{n_i} < \frac{\varepsilon}{2}$ and, since f is continuous, we may simultaneously take i such that $|f(x) - f(x_{n_i})| < \frac{\varepsilon}{2}$. But then we infer that

$$-\frac{\varepsilon}{2} < f(x) - f(x_{n_i}) < \frac{\varepsilon}{2}$$

$$\Rightarrow f(x) > f(x_{n_i}) - \frac{\varepsilon}{2} > M - \frac{1}{n_i} - \frac{\varepsilon}{2} > M - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = M - \varepsilon,$$

a contradiction. Therefore $f(x) = M$ and so that $f(x)$ attains a maximum on $[a, b]$.

9. Note that $-f(x)$ is continuous on $[a, b]$ so that by what we have just proved, $-f(x)$ attains its maximum, m say at $x \in [a, b]$ say. Then $f(x) = -m$

and we claim this is the minimum value for f on $[a, b]$ for if not, there exists some $y \in [a, b]$ such that $f(y) = p < -m$. But then $-f(y) = -p > m$, contradicting that m is the maximum value for $-f$ on $[a, b]$.

Comment The theorem represented by the pair of results of Questions 8 and 9 is called the *Extremum theorem*, in that it says that a continuous function on a bounded closed interval has extreme values (maxima and mininma).

10. Since we are assuming that $f(x) \neq M$ for all $x \in [a, b]$, it follows that $g(x)$ is continous on $[a, b]$ and so bounded (by Question 7). On the other hand since M is the least upper bound of the set of values $f(x)$ $(a \leq x \leq b)$ it follows that for any $\varepsilon > 0$ there exists $x \in [a, b]$ such that $0 \leq M - f(x) \leq \varepsilon$. But then $g(x) \geq \frac{1}{\varepsilon}$. Since ε can be taken to be arbitrarily small, we gain the contradiction that $g(x)$ is unbounded above on $[a, b]$. Therefore we conclude that for some $y \in [a, b]$, $f(y) = M$.

Problem Set 7

1. A function is continuous throughout its domain D if for any $\varepsilon > 0$ and each $a \in D$ there exists $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \varepsilon$. The value of δ here may depend on a and there is no stipulation that there is a single value of $\delta > 0$ for which this conclusion applies for all $a \in D$. However, for uniform continuity we insist that there is some $\delta > 0$ that 'works' for all $a \in D$ (although δ will still in general depend on the given value of $\varepsilon > 0$). For that reason uniform continuity is a stronger condition that continuity throughout the domain of definition of the function. That it is indeed strictly stronger is shown by the example of Question 2.

2. Let $\varepsilon > 0$ and let $a \in (0, 1]$. Take δ such that $\delta < a$. For $|x - a| < \delta$ then $a - x < \delta$ so that $0 < a - \delta < x$ and $\frac{1}{x} < \frac{1}{a-\delta}$. Hence

$$|f(x) - f(a)| = |\frac{1}{x} - \frac{1}{a}| = |\frac{a-x}{ax}| = \frac{|x-a|}{ax} < \frac{\delta}{ax} < \frac{\delta}{a(a-\delta)}$$

Now

$$\frac{\delta}{a(a-\delta)} < \varepsilon \Leftrightarrow \delta < a^2\varepsilon - \delta a\varepsilon \Leftrightarrow \delta < \frac{a^2\varepsilon}{1 + a\varepsilon}.$$

It follows that if we take $\delta < a^2\varepsilon$ then $|f(x) - f(a)| < \varepsilon$, as required to show continuity at a. Since a represents an arbitrary member of $(0, 1]$, it follows that $f(x) = \frac{1}{x}$ is continuous on $(0, 1]$.

Now put $\varepsilon = 1$. Then for any $\delta > 0$ we shall show that we may find $a \in (0, 1]$ such that there exists $x \in (0, 1]$ with $|x - a| < \delta$ but $|f(x) - f(a)| > 1$. We shall for convenience take $x < a$ and so we need x such that

$$\frac{1}{x} - \frac{1}{a} > 1 \Leftrightarrow \frac{a-x}{ax} > 1 \Leftrightarrow a - x > ax \Leftrightarrow x(a+1) > a$$

$$\Leftrightarrow x > \frac{a}{a+1}.$$

Now since $0 < \frac{a}{a+1} = 1 - \frac{1}{a+1} < 1$ we may take x to be any member of the interval $(\frac{a}{a+1}, a)$. (Note that $\frac{a}{a+1} < a$.) Therefore any value of x such that $a - \delta < x < \frac{a}{a+1}$ shows that uniform continuity fails for $\varepsilon = 1$.

3. Suppose to the contrary that $f(x)$ were not uniformly continous on $[a, b]$. Then there would exist some $\varepsilon > 0$ such that for any $\delta > 0$ there exist $x, y \in [a, b]$ such that $|x - y| < \delta$ but $|f(x) - f(y)| > \varepsilon$. Let $(\delta_n)_{n \geq 1}$ be any sequence of positive numbers monotonically decreasing to 0. Then for each δ_n there exists $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \delta_n$ but $|f(x_n) - f(y_n)| > \varepsilon$. Now since $[a, b]$ is bounded there exists a subsequence x_{n_k} of the x_n such that x_{n_k} approaches some limit x and since $[a, b]$ is closed, $x \in [a, b]$. Since f is continuous at x there is a $\eta > 0$ such that for all $y \in [a, b]$ such that $|x - y| < \eta$, implies $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Now take k sufficiently large so that so that $|x_{n_k} - x| < \eta$ and $|x - y_{n_k}| < \eta$. Then we have:

$$|f(x_{n_k}) - f(y_{n_k})| = |f(x_{n_k}) - f(x) + f(x) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x)| + |f(x) - f(y_{n_k})|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

contradicting our choice of x_{n_k} and y_{n_k}. It follows that f is uniformly continuous on $[a, b]$.

4. We note that

$$|f(x) - f_n(x)| = \frac{|x|^n}{1-x} \leq \frac{a^n}{1-a} \ \forall |x| \leq a.$$

Given $\varepsilon > 0$, choose N sufficiently large so that

$$\frac{a^N}{1-a} < \varepsilon,$$

then we have the required inequality $|f(x) - f_n(x)| < \varepsilon$ for all $n \geq N$ and all $x \in S$.

5(a) We have $f_n(x) = (n+1)(n+2)x(1-x)^n \ n = 1, 2, \cdots$, whence$f(0) = f(1) = 0$. For $0 \leq x \leq 1$ we have $0 \leq y = 1 - x \leq 1$ also and so $|f_n(x)| \leq (n+1)(n+2)y^n$. Now for any polynomial $p(n)$ we have for any $a > 1$, $\lim_{n\to\infty} \frac{p(n)}{a^n} = 0$; here we have $p(n) = (n+1)(n+2)$ and $a = \frac{1}{y}$. Hence $\lim_{n\to\infty} f_n(x) = 0$, so that $f_n \to 0$ pointwise.

(b)

$$\int_0^1 f_n(x)\,dx = (n+1)(n+2)\int_0^1 x(1-x)^n\,dx.$$

Let $F_n = \int_0^1 x(1-x)^n\,dx$. Integration by parts gives:

$$F_n = [-\frac{x(1-x)^{n+1}}{n+1}]_0^1 + \int_0^1 \frac{(1-x)^{n+1}}{n+1}\,dx = 0 - \frac{1}{(n+1)(n+2)}[(1-x)^{n+2}]_0^1$$

$$= \frac{1}{(n+1)(n+2)}.$$

It follows that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty} 1 = 1.$$

On the other hand

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

We conclude that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

(c) Suppose to the contrary that $f_n \to 0$ uniformly. Put $\varepsilon = 1$. Then there exists N such that for all $x, y \in [0,1]$ for any $n \geq N$, we have $|f_n(x) - f_n(y)| < 1$. If particular, if we put $y = 0$ we have $0 \leq f_n(x) < 1$. However, if we take $n \geq N$ and also put $x = \frac{1}{n}$ we then have

$$f_n(\frac{1}{n}) = \frac{(n+1)(n+2)(1-\frac{1}{n})^n}{n} = (1+\frac{1}{n})(1-\frac{1}{n})^n(n+2);$$

$$\lim_{n\to\infty}(1+\frac{1}{n})(1-\frac{1}{n})^n = e^{-1};$$

in particular, for all sufficiently large n we have $f_n(\frac{1}{n}) > \frac{n+2}{2e}$ and the latter increases without bound as $n \to \infty$, contrary to our choice of N. Hence no such N exists and the sequence of functions f_n does not converge uniformly to its pointwise limit of the zero function.

6(a) Let $\varepsilon > 0$. Then there exists N_1 and N_2 such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N_1$ and $|g_n(x) - g(x)| < \varepsilon$ for all $n \geq N_2$. Put $N = \max(N_1, N_2)$. Then both the previous inequalities hold for all $n \geq N$. Then for $n \geq N$ we have:

$$|af_n(x) + bg_n(x) - (af(x) + bg(x))| = |a(f_n(x) - f(x)) + b(g_n(x) - g(x))|$$

$$\leq |a||f_n(x) - f(x)| + |b||g_n(x) - g(x)| \leq |a|\varepsilon + |b|\varepsilon = (|a| + |b|)\varepsilon;$$

and since this is a fixed multiple of ε, we may connclude that $af_n+bg_n \to af+bg$ uniformly on S.

(b) Since $|f(x)|$ and $|g(x)|$ are both bounded above for all n and for all $x \in S$, there exists a common positive bound M say for both. Take $\varepsilon > 1$ in the definition of uniform convergence, as in part (a) take N such that for all $n \geq N$ we have $|f_n(x) - f(x)| < 1$ and $|g_n(x) - g(x)| < 1$, from which it follows that $|f_n(x)| < M+1$ and $|g_n(x)| < M+1$. For any $\varepsilon > 0$ we may take $N_1 \geq N$ such that for all $n \geq N_1$, $|f_n(x) - f(x)| < \varepsilon$, $|g_n(x) - g(x)| < \frac{\varepsilon}{2M+1}$. Then

$$|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)|$$

$$\leq |f_n(x)(g_n(x) - g(x))| + |g(x)(f_n(x) - f(x)| \leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|$$

$$< (M+1)\frac{\varepsilon}{2M+1} + M\frac{\varepsilon}{2M+1} = \varepsilon.$$

This establishes that $f_n g_n \to fg$ uniformly on S.

7(a) Since $|f(x)| - f(x) \geq 0$ it follows that

$$\int_a^b (|f(x)| - f(x)|)\, dx \geq 0$$

$$\Rightarrow \int_a^b |f(x)|\, dx \geq \int_a^b f(x)\, dx.$$

Replacing $f(x)$ by $-f(x)$ and noting that $|-f(x)| = |f(x)|$ we also see that $\int_a^b |f(x)|\, dx \geq \int_a^b -f(x)\, dx$. We therefore conclude that:

$$\int_a^b |f(x)|\, dx \geq |\int_a^b f(x)\, dx|.$$

(b) Take $n \geq N$, where N is chosen so that for all such n, $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$. Then we have from part (a) that:

$$|\int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx| = |\int_a^b (f_n(x) - f(x))\, dx|$$

$$\leq \int_a^b |f_n(x) - f(x)|\, dx \leq \int_a^b \frac{\varepsilon}{b-a}\, dx = (b-a)\frac{\varepsilon}{b-a} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\lim_{n\to\infty} |\int_a^b f_n(x) - \int_a^b f(x)\, dx| = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx.$$

8. Let $s_n(x)$ denote $\sum_{k=0}^{n} u_k(x)$. Then by definition, $\sum_{k=0}^{\infty} u_n(x) = \lim_{n\to\infty} s_n(x)$. We may write this as $\lim_{n\to\infty} s_n(x) = s(x)$ in which case to say that $(s_n(x))_{n\geq 0}$ converges uniformly on S means that for any $\varepsilon > 0$ there exists N such that for all $n \geq N$,

$$|s(x) - s_n(x)| = |\sum_{k=n+1}^{\infty} u_k(x)| < \varepsilon \;\forall\, x \in S.$$

If this is the case then by Question 5 we have:

$$\lim_{n\to\infty} \int_a^b \sum_{k=0}^{n} u_k(x)\, dx = \int_a^b \lim_{n\to\infty} \sum_{k=0}^{n} u_k(x)\, dx \tag{9}$$

Now, by the linearity of the integral we have:

$$\int_a^b \sum_{k=0}^{n} u_k(x)\, dx = \sum_{k=0}^{n} \int_a^b u_k(x)\, dx$$

$$\Rightarrow \lim_{n\to\infty} \int_a^b \sum_{k=0}^{n} u_k(x)\, dx = \lim_{n\to\infty} \sum_{k=0}^{n} \int_a^b u_k(x)\, dx = \sum_{k=0}^{\infty} \int_a^b u_k(x)\, dx.$$

Hence (11) becomes the required equation:

$$\sum_{k=0}^{\infty} \int_a^b u_k(x)\, dx = \int_a^b \sum_{k=0}^{\infty} u_k(x)\, dx.$$

9. By applying Question 7(b) we may change the order of the limiting operations and then by the Fundamental theorem of calculus we obtain:

$$\int_a^x g(t)\, dt = \int_a^x \lim_{n\to\infty} f_n'(x)\, dx = \lim_{n\to\infty} \int_a^x f_n'(t)\, dt = \lim_{n\to\infty} [f_n(t)]_a^{t=x}$$

$$= \lim_{n\to\infty} [f_n(x) - f_n(a)] = f(x) - f(a)$$

$$\Rightarrow f'(x) = g(x) = \lim_{n\to\infty} f_n'(x) \;\forall\, x \in [a, b].$$

10. This is a special case of the result of Question 9 where we take $f_n(x) = \sum_{k=0}^{n} u_k(x)$.

Problem Set 8

1. The sequence of functions under consideration here is the sequence of partial sums $s_n(x) = \sum_{k=0}^{n} u_k(x)$. For any $x \in S$ we have:

$$\sum_{k=0}^{\infty} |u_k(x)| \leq \sum_{k=0}^{\infty} v_k < \infty.$$

Hence the series is absolutely convergent, and so the series $\sum_{k=0}^{\infty} u_k(x)$ converges pointwise to some limiting function $u(x)$ for $x \in S$. What is more, given any $\varepsilon > 0$ we may take N such that for all $n \geq N$:

$$\sum_{k=n+1}^{\infty} v_k < \varepsilon.$$

Hence we have

$$|u(x) - s_n(x)| = |\sum_{k=n+1}^{\infty} u_k(x)| \leq \sum_{k=n+1}^{\infty} |u_k(x)| \leq \sum_{k=n+1}^{\infty} v_k < \varepsilon,$$

thus showing that the partial sums $s_n(x)$ converge uniformly to the limiting sum $u(x)$ on S.

2(a) Applying the ratio test:

$$|\frac{u_{k+1}}{u_k}| = \frac{a^{2(k+1)+1}k!}{a^{2k+1}(k+1)!} = \frac{a^2}{k+1} \to 0 \text{ as } k \to \infty,$$

so the series $\sum_{k=0}^{\infty} \frac{a^{2k+1}}{k!}$ converges.

(b) Putting $x = -t^2$ in the exponential series (which converges for all values of x) we have

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}.$$

Hence

$$\int_0^x e^{-t^2}\, dt = \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}\, dt.$$

If we interchange the taking of the two limits on the right hand side we are led to required conclusion:

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^x t^{2k}\, dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}[t^{2k+1}]_0^x$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}.$$

The exchange of the order of summation and integration is justified using the result of Question 1 providing we show that the series is uniformly convergent, and we do this using the Weierstrasss M-test as follows. Here we have for any x such that $|x| \leq a$ that

$$|u_k(x)| = \frac{x^{2k+1}}{(2k+1)k!} \leq \frac{a^{2k+1}}{k!};$$

if we put $v_k = \frac{a^{2k+1}}{k!}$ we have that by (a) that $\sum_{k=0}^{\infty} v_k$ converges. Therefore, by the M-test, the series $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}$ converges uniformly on every finite interval of the real line. In particular, taking $a > x$ justifies the exchange of integral and summation used above.

3. Since $\sum_{n=0}^{\infty} a_n r^n$ converges it follows that $|a_n r^n| \to 0$ as $n \to \infty$ so that the sequence $|a_n r^n| < M$ for some upper bound M. Then for any x such that $|x| < |r|$ put $|\frac{x}{r}| = \rho < 1$. Then

$$|a_n x^n| = |a_n r^n||\frac{x}{r}|^n < M\rho^n.$$

Now $\sum_{n=0}^{\infty} M\rho^n$ converges (to the limit $\frac{M}{1-\rho}$) so it follows from the comparison test that $\sum_{n=0}^{\infty} |a_n x^n|$ converges, which is to say our original series converges absolutely for $x \in (-r, r)$.

4. Certainly $f(0) = a_0$ in all cases. Suppose that $f(r)$ converges for some $r \neq 0$. By Question 1 we have that $f(x)$ converges for all x such that $|x| < |r|$. Consider the non-empty set $C = \{r : f(x)$ converges for all $x : |x| < r\}$. If C has no upper bound then for any $x \geq 0$ we could choose $r > c$ such that $x \in C$. It would then follow that $C = \mathbb{R}^+$ and since absolute convergence implies convergence, it would follow that $f(x)$ were defined for all $x \in \mathbb{R}$. In this case we say that $R = \infty$.

Otherwise we may let R be the least upper bound of the set C. Then $R \geq |r| > 0$. Take any $x_0 \in (0, R)$, but suppose that $x_0 \notin C$. Then by definition of R it follows that x_0 is not an upper bound of C so there exists $r \in C$ such that $x_0 < r$; moreover $r \leq R$ as R is an upper bound of C. But then by Question 1 we have $x_0 \in C$, contradicting our choice of x_0. Hence no such x_0 exists and so $f(x)$ converges for all $x \in (0, R)$. If $-R < x < 0$ then $-x \in C$, whence it follows that $f(x)$ also converges (again as asbolute convergence implies convergence). Therefore $f(x)$ converges for all $x \in (-R, R)$.

On the other hand, for any x_0 such that $|x_0| > R$ we have $f(x_0)$ does not converge for if $f(x_0)$ were defined, then by Question 1, the same would be true of any $y \in (R, |x_0|)$, whereupon $|x_0| \in C$, contrary to the definition of R. Therefore the series diverges for all $x > R$ or $x < -R$.

5. Let

$$r = \lim_{n\to\infty} |\frac{a_{n+1}x^{n+1}}{a_n x^n}| = |x| \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|.$$

By the ratio test the series $f(x)$ converges if $0 \leq r < 1$, which is to say that $\frac{|x|}{R} < 1$; that is if $-R < x < R$, and $f(x)$ diverges if $r > 1$, which is to say that $|x| > R$. Therefore R is indeed the radius of convergence of $f(x)$.

6. We have from Question 2 that $f(x)$ converges absolutely on $(-R, R)$ and

so does the same on $[-r, r]$. Choose any number x with $|x| < r$. Then

$$|a_n x^n| = |a_n||x|^n < |a_n|r^n = |a_n r^n|.$$

Put $M_n = |a_n r^n|$ so then $\sum_{n=0}^{\infty} M_n < \infty$. We then have by the Weierstrass M-test that $f(x)$ converges uniformly on $[-r, r]$.

7. We are considering

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n.$$

Substitute $y = x - a$ so that we have a series centred at 0, that being $g(y) = \sum_{n=0}^{\infty} a_n y^n$. By Question 6 the series is uniformly convergent on $(-r, r)$ for any $0 \le r < R$, where R is the radius of convergence of $g(y)$. Therefore $f(x)$ converges uniformly on each interval of the form

$$-r < x - a < r \Leftrightarrow -r + a < x < r + a.$$

8. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all $0 \le x < r$. Fix an x_0 with $0 < |x_0| < r$, and choose x such that $|x| < |x_0| < r$. The series $\sum_{n=0}^{\infty} a_n x_0^n$ converges and therefore $\lim_{n\to\infty} a_n x_0^n = 0$. We can thus find a number bound M such that $|a_n x_0^n| < M$ for all n. We now write

$$n a_n x^{n-1} = n a_n x^{n-1} \frac{x_0^n}{x_0^n} = \frac{n a_n}{x_0} x_0^n \big(\frac{x}{x_0}\big)^{n-1}.$$

Putting $\rho = \frac{x}{x_0} < 1$ we have

$$|n a_n x^{n-1}| = |\frac{n a_n}{x_0} x_0^n \big(\frac{x}{x_0}\big)^{n-1}| = |\frac{n a_n}{x_0} x_0^n \rho^{n-1}| \le M \frac{n}{|x_0|} \rho^{n-1}.$$

Now $\sum_{n=0}^{\infty} n\rho^{n-1}$ converges by the ratio test as the associated quotient limit is:

$$\lim_{n\to\infty} |\frac{(n+1)\rho^n}{n\rho^{n-1}}| = \lim_{n\to\infty} (1 + \frac{1}{n})\rho = \rho < 1.$$

Therefore, by the M-test, the original series converges uniformly for all x such that $|x| < |x_0|$. Since x_0 was an arbitrary number sastisfying $|x_0| < r$, the series converges uniformly for $|x| < R$. It follows that the radius of convergence R_0 of the series of derivatives satisfies $R_0 > R$, the radius of convergence of our original series. If $R = \infty$, then $R_0 = \infty$.

If $R_0 > R$ we may choose $r_0 > R$ and x such that $R < |x| < r_0$. Then, evaluated at x, the series of derivatives is absolutely convergent, while the original series diverges. But, then for all n such that $|\frac{x}{n}| < 1$ we have:

$$|a_n x^n| = |n a_n x^{n-1}||\frac{x}{n}| \le |n a_n x^{n-1}|.$$

This means that the original series converges upon substituting this value x, which is false. Therefore we may conclude that $R = R_0$, as required.

Comment From which it follows that a series that results by term-by-term integration also has the same radius of convergence as the original series.

9(a)

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \ \forall\, |x| < 1.$$

Integrating both sides now gives

$$\int \frac{dx}{1+x} = \sum_{n=0}^{\infty} \int (-1)^n x^n \, dx$$

$$\Rightarrow \log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \ \forall\, |x| < 1;$$

although we should not neglect the integration constant: however both sides agree when $x = 0$ so the integration constant is 0.

(b) Replacing x by $-x$ (noting that $|x| < 1$ if and only if $|-x| < 1$) we obtain:

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$$

10. We have

$$f(x) = \frac{x^2}{2} - \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3} - \frac{x^5}{5 \cdot 4} + \cdots \ |x| < 1$$

$$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}.$$

We apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} n(n-1)}{(-1)^n n(n+1)}\right| = \frac{n-1}{n+1} = 1 - \frac{2}{n+1} \to 1.$$

Hence the radius of convergence of $f(x)$ is 1. Differentiating term-by-term gives:

$$f'(x) = \sum_{n=2}^{\infty} \frac{(-1)^n x^{n-1}}{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = \log(1+x).$$

Hence

$$f(x) = \int \log(1+x)\, dx.$$

We integrate by parts with $u = \log(1+x)$, $dv = dx$ so that $du = \frac{dx}{1+x}$ and $v = x$ to give

$$f(x) = x\log(1+x) - \int \frac{x}{1+x}\,dx = x\log(1+x) - \int (1 - \frac{1}{1+x})\,dx$$

$$= x\log(1+x) - x + \log(1+x) + c$$

$$\Rightarrow f(x) = (1+x)\log(1+x) - x + c.$$

Put $x = 0$ we get $f(0) = 0 = 0 + c$ so that $c = 0$ and indeed

$$f(x) = (1+x)\log(1+x) - x$$

Problem Set 9

1. Since f is continuous on $[a, b]$ it takes on both a minimum m and a maximum value M on $[a, b]$. If these values occur at the endpoints a and b, then since $f(a) = f(b)$, it follows that f is a constant function and we may take c to be any member of (a, b) in order to satisfy the conclusion of the theorem. Otherwise one of these extrema, let us say M occurs at some point $c \in (a, b)$. Then we have for $h > 0$:

$$\frac{f(c+h) - f(c)}{h} = \frac{f(c+h) - M}{h} \leq 0$$

$$\Rightarrow f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - M}{h} \leq 0 \tag{10}$$

For $h < 0$ the calculation is the same except for the change of sign in the denominator giving:

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - M}{h} \geq 0 \tag{11}$$

It follows from (12) and (13) that $f'(c) = 0$, as required.

Comment According to Wikipedia the Indian mathematician Bhāskara II (1114–1185) is credited with knowledge of Rolle's theorem, although the theorem is named after Michel Rolle. Rolle's 1691 proof covered only the case of polynomial functions. His proof did not use the methods of differential calculus, which at that point in his life he considered to be fallacious. The theorem was first proved by Cauchy in 1823 as a corollary of a proof of the mean value theorem.

2. Let $[a, b] = [-1, 1]$ and $f(x) = |x|$. Then $f(x)$ is continuous on $[a, b]$ with $f'(x) = -1$ if $x \in [-1, 0)$, $f'(x) = 1$ if $x \in (0, 1]$ but $f'(0)$ is not defined (as the

corresponding limit is ± 1 according as $h \to 0^-$ or $h \to 0^+$.) Therefore Rolle's theorem does not generally hold if $f(x)$ is not differentiable at some point in (a, b).

3(a) Define $g(x) = f(x) - rx$. Put $g(a) = f(a) - ra = g(b) = f(b) - rb$. Solving for r then gives $r(b-a) = f(b) - f(a)$ so that

$$r = \frac{f(b) - f(a)}{b - a}.$$

(b) Clearly $g(x)$ is also continuous on $[a, b]$ and differentiable on (a, b) and what is more, with the choice of the constant r as in part (a), $g(x)$ satisfies $g(a) = g(b)$ and so Rolle's theorem applies to $g(x)$. Hence there exists $c \in (a, b)$ for which $g'(c) = 0$, which is to say that$f'(c) - r = 0$ whence

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

4(a) We have $P(a) = f(a)$ so the claim holds for $k = 0$. When differentiating $P_n(x)$ k times, $(k \leq n)$ all terms involving $(x - a)^m$ with $m \leq k - 1$ vanish. The kth derivative of $\frac{f^{(k)}(a)}{k!}(x - a)^k$ is the constant term $f^{(k)}(a)$ while the kth derivative of powers of $x - a$ higher than k each takes the value 0 under the substitution $x \mapsto a$. Therefore $P_n^{(k)}(a) = f^{(k)}(a)$ for all $k = 0, 1, \cdots, n$.

(b)

$$F(x) = f(b) - f(x) - f'(x)(b - x) - \frac{f^{(2)}(x)}{2!}(b - x)^2 - \cdots - \frac{f^{(n-1)}(x)}{(n-1)!}(b - x)^{n-1}; \tag{12}$$

the contribution to $F'(x)$ from the entry $-\frac{f^{(k)}(x)}{k!}(b - x)^k$ is

$$-\frac{f^{(k+1)}(x)}{k!}(b - x)^k + \frac{f^{(k)}(x)}{(k-1)!}(b - x)^{k-1};$$

the second term here cancels the same term with a negative sign in the previous entry. It follows that the full expression for $F'(x)$ telescopes down with the only remaining contribution being $-\frac{f^{(n)}(x)}{(n-1)!}(b - x)^{n-1}$, as required.

(c)

$$g(x) = F(x) - \Big(\frac{b - x}{b - a}\Big)^n F(a);$$

clearly $g(b) = F(b) = 0$ and $g(a) = F(a) - F(a) = 0$. Moreover, since $F(x)$ may be differentiated on (a, b) at least once, the same is true of $g(x)$. Hence we may apply Rolle's theorem to $g(x)$ to conclude that there exists $c \in (a, b)$ such that $g'(c) = 0$, which is to say that

$$F'(c) + n\frac{(b - c)^{n-1}}{(b - a)^n}F(a) = 0$$

$$\Rightarrow -\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1} = -\frac{n(b-c)^{n-1}}{(b-a)^n}F(a)$$

$$\Rightarrow F(a) = \frac{f^{(n)}(c)(b-a)^n}{n!}$$

We may now put $x = a$ in (14) to obtain Taylor's theorem:

$$f(b) = f(a)+f'(a)+\frac{f^{(2)}(a)}{2!}(b-a)^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+\frac{f^{(n)}(c)(b-a)^n}{n!}.$$

5. Put $f(x) = \cos x$ so that $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$. We consider the Taylor series for $f(x)$ about $a = 0$; $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$. By Taylor's theorem there exist c between 0 and x such that

$$\cos x = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(c)x^3}{3!} = 1 - \frac{1}{2}x^2 - \frac{(\sin c)x^3}{6}.$$

For $|x| \le \pi$ observe that $(\sin c)x^3 \ge 0$. On the other hand if $|x| \ge \pi$ then $1-\frac{1}{2}x^2 < -3 < \cos x$. Therefore for all values of x we have that $\cos x \le 1-\frac{1}{2}x^2$.

6(a) Apply Taylor's theorem. In this case the fact that $f^{(k)}(x_0) = 0$ for all $k = 1, 2, \cdots, n-1$ means all the corresponding terms vanish and we are left with

$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n \tag{13}$$

for some c in the open interval with endpoints x_0 and x.

(b) Suppose now that $f^{(n)}(x_0) > 0$ and n is even, so that $(x-x_0)^n \ge 0$. Since $f^{(n)}(x)$ is continuous at x_0 there is an open interval I containing x_0 such that $f^{(n)}(x) > 0$ for all $x \in I$. By Taylor's theorem, the equation (15) holds for some $c \in I$ from which it follows that $f(x) \ge f(x_0)$ for all $x \in I$ so that x_0 is a local minimum of $f(x)$.

7. We take $f(x) = \log(1+x)$, $f'(x) = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f^{(3)}(x) = 2(1+x)^{-3}, \cdots, f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$. We apply Taylor's theorem with $a = 0$ in which case $f(a) = 0$ and

$$\frac{f^{(n)}(a)}{n!} = \frac{(-1)^{n-1}}{n};$$

hence by Taylor's theorem, for any $x > 0$ there exists $c \in (1, 1+x)$ such that

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{(n+1)(1+c)^{n+1}}x^{n+1}.$$

The final remainder term is positive if n is even and negative if n is odd. Therefore we obtain:

$$x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots-\frac{1}{2k}x^{2k} < \log(1+x) < x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots+\frac{1}{2k+1}x^{2k+1}.$$

8(a) We work with the case of $g(x) \geq 0$, with the $g(x) \leq 0$ case very similar. By the Extreme value theorem there exists bounds m and M for $f(x)$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$ and f attains these bounds (which is important in part (b)). Hence it follows that

$$m \int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M \int_a^b g(x)\,dx. \tag{14}$$

If $\int_a^b g(x)\,dx = 0$ then it follows from (16) that $\int_a^b f(x)g(x)\,dx = 0$ also and we may take any $c \in [a, b]$ to satisfy the conclusion of the theorem. Otherwise we may divide to obtain:

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M.$$

(b) Since $f(x)$ attains the bounds of m and M in the interval $[a, b]$ it follows by the Intemediate value theorem that there exists $c \in [a, b]$ such that $f(c) = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$, whence we gain the required conclusion that:

$$\int_a^b f(x)g(x)\,dx = f(c) \int_a^b g(x)\,dx.$$

9(a)

$$A = \phi(x + h) - \phi(x), \text{ where } \phi(x) = f(x, y + k) - f(x, y).$$

Since ϕ is differentiable with respect to x (keeping k and y fixed) we may invoke the MVT on the interval $[a, a + h]$ to conclude that

$$\frac{A}{(x + h) - x} = \phi'(x + \theta h)$$

$$\Rightarrow A = h\phi'(x + \theta h)$$

for some $0 < \theta < 1$.

(b) Now

$$\phi'(x) = f_x(x, y + k) - f_x(x, y),$$

and since the mixed partial derivative f_{yx} exists we may apply the MVT to the function defined by the expression on the right as a function of y to conclude, again by the MVT , that

$$\frac{A}{h} = \frac{f_x(x + \theta h, y + k) - f_x(x + \theta h, y)}{(y + k) - y} = f_{yx}(x + \theta h, y + \theta' k)$$

$$\Rightarrow A = hkf_{yx}(x + \theta h, y + \theta' k), \tag{15}$$

where $0 < \theta, \theta' < 1$.

(c) By interchanging the roles of x and y in the the previous argument, (beginning with the function $\psi(y) = f(x+h,y) - f(x,y)$), we may likewise conclude that

$$A = hkf_{xy}(x + \theta_1 h, y + \theta_1' k) \tag{16}$$

Equation the two expressions for A from (17) and (18) we have

$$A = f_{yx}(x + \theta h, y + \theta' k) = f_{xy}(x + \theta_1 h, y + \theta_1' k);$$

we now let $h, k \to 0$, whence by the assumed continuity of f_{yx} and f_{xy} at (x, y) we conclude that $f_{yx}(x,y) = f_{xy}(x,y)$.

Comment Many important theorems in calculus come down to equality being maintained when the order of two limiting operations is reversed. The proofs often depend on the Mean value theorem. Equality of mixed partial derivatives is a key example as it is assumed in many of the big theorems of Vector analysis such as Green's theorem and Stokes Theorem in its various forms.

10(a)

$$f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

$$f_x(0,y) = \lim_{x\to 0} \frac{f(x,y) - f(0,y)}{x} = \lim_{x\to 0} y\frac{x^2 - y^2}{x^2 + y^2} = -y;$$

$$f_y(x,0) = \lim_{y\to 0} \frac{f(x,y) - f(x,0)}{y} = \lim_{y\to 0} x\frac{x^2 - y^2}{x^2 + y^2} = x.$$

(b) Consequently $f_{xy}(x,0) = 1$ and $f_{yx}(0,y) = -1$. In particular $f_{xy}(0,0) = 1 \neq -1 = f_{yx}(0,0)$.

Problem Set 10

1. We have $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Since $f(x)$ may be differentiated term-by-term and the resulting series have the same radius of convergence it follows that the constant term of the series expansion of $f^{(n)}(x)$ that results is $n!a_n$. Putting $x = 0$ now gives

$$a_n = \frac{f^{(n)}(0)}{n!} \;\forall\, n = 0, 1, 2, \cdots.$$

Comment We conclude that a smooth function cannot have two different series expansions about the same centre. Hence if we arrive at the series in two different ways, we may use equating of coefficients to assist in the determination of those coefficients. This is the basis of justification for finding the series for functions that result from several series combined using arithmetic operations (linear combinations, multiplication and division) and composition (substitution). However, a smooth function does not necessarily have a convergent Taylor series. For example, it may be shown that $f^{(k)}(0) = 0$ for all $k \geq 0$ for the function defined by the rule $f(x) = e^{-\frac{1}{x^2}}$ (with $f(0) = 0$). The resulting Taylor series is shared with the zero function and clearly converges to the latter and not the former. Despite being 'completely flat' at the origin, this function manages to pick itself up off the real line away from zero. This baffling behaviour is partly explained when we extend the function to a complex variable as there we find infinitely many singularities in every neighbourhood of the origin, although none on the real line itself.

2. $R_1(x) = f(x) - P_1(x) = f(x) - f(a) - f'(a)(x - a)$. We check this agrees with the integral, which for $n = 1$ is:

$$I_1(x) = \frac{1}{1!}\int_a^x (x - t)f^{(2)}(t)\,dt.$$

Put $u = x - t$ and $dv = f^{(2)}(t)\,dt$,so $du = -dt$ and $v = f'(t)$:

$$I_1(x) = (x - t)f'(t)]_{t=a}^{t=x} + \int_a^x f'(t)\,dt$$

$$= 0 - (x - a)f'(a) + f(x) - f(a) = f(x) - f(a) - f'(a)(x - a).$$

3. Now we assume inductively that for some $n = k$ we have:

$$R_k(x) = \frac{1}{k!}\int_a^x (x - t)^k f^{(k+1)}(t)\,dt$$

and consider

$$I_{k+1} = \frac{1}{(k+1)!}\int_a^x (x - t)^{k+1} f^{(k+2)}(t)\,dt.$$

Put $u = (x - t)^{k+1}$ so $du = -(k + 1)(x - t)^k\,dt$ and $dv = f^{(k+2)}(t)$ so that $v = f^{(k+1)}(t)$. We obtain upon integrating by parts in this way:

$$I_{k+1} = \frac{1}{(k+1)!}(x - t)^{k+1} f^{(k+1)}(t)]_{t=a}^{t=x} + \frac{k+1}{(k+1)!}\int_a^x (x - t)^k f^{(k+1)}(t)\,dt$$

$$= 0 - \frac{1}{(k+1)!}(x - a)^{k+1} f^{(k+1)}(a) + \frac{1}{k!}\int_a^x (x - t)^k f^{(k+1)}(t)\,dt$$

$$= -\frac{f^{(k+1)}(a)}{(k+1)!}(x - a)^{k+1} + R_k(x)$$

$$= f(x) - P_k(x) - \frac{f^{(k+1)}(a)}{(k+1)!}(x - a)^{k+1}$$

$$= f(x) - P_{k+1}(x) = R_{k+1}(x).$$

4. With $a = 0$ the remainder term has the form:

$$R_n(x) = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,dt;$$

for $x > 0$ we have

$$|R_n(x)| \le \frac{1}{n!}\int_0^x (x-t)^n |f^{(n+1)}(t)|\,dt.$$

In the case of $f(x) = \sin x$ we have that $-1 \le f^{(n)}(t) \le 1$ so this simplifies to

$$|R_n(x)| \le \frac{1}{n!}\int_0^x (x-t)^n\,dt = \frac{1}{n!}\frac{x^{n+1}}{(n+1)} = \frac{x^{n+1}}{(n+1)!}.$$

For $x < 0$ we have

$$R_n(x) = -\frac{1}{n!}\int_x^0 |x-t|^n |f^{(n+1)}(t)|\,dt$$

$$\Rightarrow |R_n(x)| \le \frac{1}{n!}\int_x^0 (t-x)^n\,dt = \frac{(-x)^{n+1}}{(n+1)!}.$$

Therefore in either case we may conclude that

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

We conclude that $\sin x$ is equal to the sum of its McLaurin series for all $x \in \mathbb{R}$.

5(a) $f(x) = x^{\frac{1}{3}}$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$,$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$, $f^{(3)}(x) = \frac{10}{27}x^{-\frac{8}{3}}$; $f(8) = 2$, $f'(8) = \frac{1}{12}$, $f''(8) = -\frac{1}{144}$. Hence we obtain:

$$\begin{aligned} P_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

(b) Using the Lagrange form of the remainder we have that for some c with $7 \le c \le 8$:

$$R_2(7) = \frac{f^{(3)}(c)}{3!}(7-8)^2 = \frac{1}{6}\cdot\frac{10}{27}c^{-\frac{8}{3}} = \frac{5}{81c^{\frac{8}{3}}}.$$

Since we are looking for an (upper) bound on $|R_2(x)|$ we maximize this quantity by taking c to be as small as possible, so we conclude that

$$|R_2(7)| \le \frac{5}{81\cdot 7^{\frac{8}{3}}} < 0\cdot 0004.$$

6(a) By replacing x by $-x^2$ in the exponential series we obtain:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots.$$

(b)

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 (1 - x^2 + \frac{x^4}{2})\,dx = [x - \frac{x^3}{3} + \frac{x^5}{10}]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} = \frac{23}{30} \approx 0\cdot 766.$$

Since the series has alternating signs and the terms approach 0 monotonically (for $0 < x < 1$), the remainder is bounded by the magnitude of the next term and the product of the length of the interval of integration (1 in this case). Since the next term is negative, our answer is an over-estimate and the error in the approximation is no more than the maximum of the next term in the integration, which is

$$\frac{1^7}{7 \cdot 3!} = \frac{1}{42} \approx 0 \cdot 024.$$

7.

$$x^2 + y^2 = y,\ y(0) = 1$$

$$\Rightarrow 2x + 2yy' = y' \Rightarrow y'(1 - 2y) = 2x \Rightarrow y' = \frac{2x}{1 - 2y}$$

$$\Rightarrow y'(0) = \frac{0}{1 - 2} = 0;$$

$$y'' = \frac{2(1 - 2y) + 2yy'}{(1 - 2y)^2} \Rightarrow y''(0) = \frac{2(1 - 2) + 2(1)(0)}{(1 - 2)^2} = \frac{-2}{(-1)^2} = -2.$$

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \cdots = 1 + 0 + \frac{-2}{2}x^2 + \cdots$$

$$\therefore y(x) = 1 - x^2 + \cdots.$$

8.

$$f(x, y) = \sum_{m,n=0}^{\infty} a_{m,n}(x - a)^m (y - b)^n$$

$$\Rightarrow \frac{\partial f^{m+n}}{\partial x^m \partial y^n} = m!n!a_{m,n} + \text{non-constant terms in powers of } x \text{ and } y$$

$$\Rightarrow \frac{\partial f^{m+n}}{\partial^m x \partial^n y}|_{(x,y)=(a,b)} = m!n!a_{m,n}$$

$$\therefore a_{m,n} = \frac{1}{m!n!}\frac{\partial f^{m+n}}{\partial^m x \partial^n y}|_{(x,y)=(a,b)} \qquad (17)$$

9(a) Using subscript notation for partial derivatives, the linear approximating polynomial in x and y involves all terms as in (19) with $m + n \leq 1$ giving

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

which is the equation of the tangent plane to the surface $z = f(x, y)$ at the point (a, b).

(b) As in (a) but now we proceed with constraint $m + n \leq 2$.

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$+\frac{1}{2} f_{xx}(a, b)(x - a)^2 + \frac{1}{2} f_{yy}(a, b)(y - b)^2 + f_{xy}(a, b)(x - a)(y - b).$$

10. $f(x, y) = (1 - x - y)^{-1}$,

$$\frac{\partial f^{m+n}}{\partial^m x \partial^n y}(x, y) = (m + n)!(1 - x - y)^{-(m+n+1)}$$

$$\Rightarrow \frac{\partial f^{m+n}}{\partial^m x \partial^n y}|_{(x,y)=(0,0)} = (m + n)!$$

$$\therefore \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m + n)!}{m!n!} x^m y^n$$

represents the Taylor series expanded about the origin for $(1 - x - y)^{-1}$.